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FUZZY IDENTITY

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Abstract

After the introductory discussion, fuzzy congruencies were introduced as fuzzy relations by fuzzy subalgebras of crisp algebra. Fuzzy subalgebras relate to fuzzy congruencies via a suitable definition of reflexivity. It has been proven that the collection of all fuzzy congruencies on a fuzzy subalgebra of some algebra is a complete lattice in which the fuzzy equalities form a complete i-semi-lattice. Fuzzy identity is further defined as a formula where the terms are related via fuzzy equality instead of crisp. A fuzzy identity can be satisfied on a fuzzy subalgebra (with respect to some fuzzy equality), while the crisp support algebra does not have to conform to the corresponding crisp identity. It is proved that, if the fuzzy subalgebra of the algebra satisfies the fuzzy identity relative to some fuzzy equality, then there exists a smallest fuzzy equality such that the corresponding fuzzy identity holds on the same fuzzy subalgebra. The main result in this work is that a crisp algebra satisfies a crisp identity if and only if it is appropriate a fuzzy subalgebra satisfies the corresponding identity for all corresponding fuzzy equalites.

Key words: *Fuzzy congruency, fuzzy subalgebra, fuzzy identity, fuzzy equality.*

Introduction

Fuzzy versions of particular algebraic structures and notions in universal algebra are important topics in theoretical investigations of fuzziness. Indeed, all these relay on substructures which in the fuzzy framework depend of membership function; as it is known, the latter offers much more possibilities then the classical characteristic function. Chronologically, first there were fuzzy groups (Rosenfeld [17] and Das [6]), then semigroups, rings and other particular structures and applications (see Malik and Mordeson e.g., [16] and the corresponding references). Investigations of notions from universal algebra followed (see e.g., Di Nola, Gerla [10] and [20, 21]). In all these the universe of an algebra was fuzzified, while the operations remained crisp. The set of values was different: either it was the unit interval, or a complete lattice. In the last period (10 – 20 years) this lattice is mostly residuated. Generalized co-domains are

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also used, like particular posets or relational systems ([20]). In the mentioned last two decades most topic from universal algebra have been systematically investigated. We here mention only some particular algebraic contributions, relevant to the present investigation. The notion of fuzzy equality was introduced by H¨ohle ([13]) and then used by many others. In several papers, see e.g. [7, 8], Demirci considers particular algebraic structures equipped with fuzzy equality relation; he also uses compatible fuzzy functions, without switching to universal algebras. Bˇelohl´avek (see his book [1], references there, and recent paper with Vychodil [2]) introduces and investigates algebras with fuzzy equalities. These are defined as classical algebras in which crisp equality is replaced by a fuzzy one being compatible with the fundamental operations of the algebra. In this fuzzy framework, Bˇelohl´avek develops and investigates the most important universal algebraic topics (congruences, subalgebras, products, and also varieties). The set of values of his fuzzy structures is usually a complete residuated lattice.

The present paper deals with fuzzy congruences on fuzzy subalgebras of a crisp algebra. Special fuzzy congruences are fuzzy equalities and these are introduced instead of the ordinary crisp equality. The set of values is a complete residuated lattice, which in some cases possesses additional properties.

After the preliminary Section 1, in Section 2 we introduce fuzzy congruence as fuzzy relations on fuzzy subalgebras of a crisp algebra. Fuzzy congruences are related to fuzzy subalgebras by suitably defined reflexivity.We prove that the collection of all fuzzy congruences on a fuzzy subalgebra of an algebra is a complete lattice in which fuzzy equalities form a complete meet-semilattice.

In Section 3, we define fuzzy identities on a fuzzy subalgebra, as formulas in which terms are related by fuzzy equalities instead of the crisp one. Fuzzy identity may be satisfied by a fuzzy subalgebra (with respect to some fuzzy equality), while the underlying crisp algebra need not satisfy the analogue crisp identity. We were able to prove that if a fuzzy subalgebra of an algebra satisfies a fuzzy identity with respect to some fuzzy equality, then there is a least fuzzy equality such that the corresponding fuzzy equality holds on the same fuzzy subalgebra. Our main result in this section is that a crisp algebra fulfils a crisp identity if and only if a particular fuzzy subalgebra satisfies the analogue fuzzy identity for all corresponding fuzzy equalities.

Section 4 deals with special fuzzy congruences on a crisp algebra, so called *weak* fuzzy congruences. Under particular conditions, we prove that every fuzzy weak congruence on an algebra is a fuzzy congruence on a fuzzy subalgebra; conversely, every fuzzy congruence on a fuzzy subalgebra of an algebra is a weak fuzzy congruence on the same crisp algebra.

Preliminaries

Algebras

We advance some notions from universal algebra, together with their relevant properties; more can be found e.g., in the book [4].

A **language** or a **type** of algebras is a set ***F*** of functional symbols, together with a set of natural numbers with zero (arities) associated to these symbols. An **algebra** is a pair (***A, F***), denoted by ***A***, where ***A*** is a nonempty set and ***F*** is a set of (fundamental) operations on ***A***. Each operation in F corresponds to some symbol in the language; if it is n-ary, then its arity is n. A **subalgebra** of **A** is an algebra of the same type, efined on a non-empty subset of ***A***. A subset closed under fundamental operations is called a **subuniverse** of **A**, which can also be the empty set. **Terms** in the language of the algebra are usual regular expressions constructed by the variables and operational symbols (see [4] for the precise definition). An **identity** in the language of **A** is a formula $t_1 = t_2$, where t_1, t_2 are terms in the same language. An equivalence relation ρ on ***A*** which is compatible with respect to all fundamental operations ($x_i \rho y_i$, $i = 1, \ldots,$ n imply $f(x_1, \ldots, x_n) \rho f(y_1, \ldots, y_n)$) is a **congruence** on **A**. The quotient set A/ρ, where ρ is a **congruence** on **A** is a **quotient algebra** of the same type.
In the context of our present investigations, some relevant topics from universal algebra can be found in the book [18].

Residuated lattices

Recall that a poset (***L*, ≤**) in which for every two-element subset $\{x, y\}$ there exist the infimum ($x \wedge y$, **meet**) and the supremum ($x \vee y$, **join**) is a **lattice**. It is **complete** if the infimum and the supremum exist for every subset of ***L***. As an algebraic structure, a complete lattice is usually denoted by (L, ∧, ∨, 0, 1), where 0 and 1 are the smallest and the greatest element under the order ≤ in ***L***. Further on, a **commutative monoid** (L, ⊗, 1) is a nonempty set L together with a commutative and associative binary operation ⊗ and a unit (neutral) element $1 \in L$ (for every $x \in L$, $1 \otimes x = x \otimes 1 = x$). For detailed presentation of lattice and order theory see e.g. [5]).
A **complete residuated lattice** is here considered to be a structure
$(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ in which

- $(L, \wedge, \vee, 0, 1)$ is a *complete lattice*,
- $(L, \otimes, 1)$ is a *commutative monoid*, and

the binary operations ⊗ and → form an *adjoint pair*, meaning that for all $x, y, z \in L$,
$x \otimes y \le z$ if and only if $x \le y \rightarrow z$.

Residuated lattices were introduced in 1939. by Dilworth and Ward ([9]). Recently, Blount and Tsinakis investigated these algebras in a more general way ([3]). A detailed study of residuated lattices can be found in a recent book by Galatos et al. ([11]).
From the point of view of fuzzy mathematics, an extensive presentation of residuated lattices is given in a book by Bˇelohl´avek ([1]). Here we deal with fuzzy relations, hence we use terminology, notions and related properties from this book. Apart from general lattice properties (see eg., [5]), concerning additional operations we use frequently the fact that *multiplication ⊗ is isotone with respect to the lattice order ≤ and*

distributive with respect to arbitrary suprima. Observe also that 0 *is absorbing with respect to multiplication*: $0 \otimes x = 0$, for every x.
There are many examples of complete residuated lattices whose lattice reduct (i.e., its part which is a complete lattice) is the unit interval of the real line, together with min and max as the lattice operations. Depending how the other two binary operations are defined ($\otimes$ and $\to$), there are *L- ukasiewicz structure*, *G¨odel structure*, *Product structure* and others. All these are linearly ordered, but there are others which are not, like Heyting and Boolean lattices, both under suitably defined adjoint operations (for these, see also [1]).
We say that a residuated lattice L is **zero divisor free** if $x \otimes y = 0$ imply $x = 0$ or $y = 0$.
Let us point out that residuated lattices are a subclass of *lattice ordered monoids* (see e.g., [15]), which could also serve as a co-domain of values for fuzzy sets and relations in our investigation.
We need also the following order-theoretic notions. An ordered set ($L, \leq$) is a **meet-semilattice** if every two-element subset contains the greatest lower bound (meet); dually, (L, $\leq$) is a **join-semilattice** if every twoelement subset has the least upper bound (join). Equivalently, in terms of algebras, a semi-lattice is an *idempotent, commutative semigroup*. A **semi-ideal** in an ordered set (P, $\leq$) is its subset I fulfilling: for every $x \in P$ and $y \in I$, if $x \leq y$ then $x \in I$.

Fuzzy sets, structures and relations

Let (L, $\wedge$, $\vee$, $\otimes$, $\to$, 0, 1) be a complete residuated lattice.
A fuzzy sets μ on a nonempty set A is a function $\mu : A \to L$.
For $p \in L$, a **cut set**, or a p-**cut** (sometimes simply a **cut**) of μ is a subset μ_p of X which is the inverse image of the principal ideal in L, generated by p:

$$\mu_p = \{x \in X \mid \mu(x) \geq p\}.$$

A mapping $\rho : A^2 \to L$ is a fuzzy relation on A. Some well known properties of fuzzy relations:

ρ is **reflexive** if $\rho(x, x) = 1$ for all $x \in A$; (1)
ρ is **symmetric** if $\rho(x, y) = \rho(y, x)$ for all $x, y \in A$; (2)
ρ is **transitive** if $\rho(x, y) \geq \rho(x, z) \otimes \rho(z, y)$ for all $x, y, z \in A$. (3)

A fuzzy relation ρ on A which is reflexive symmetric and transitive is a **fuzzy equivalence** on A. If, in addition ρ fulfills also the property

$\rho(x, y) = 1$ if and only if $x = y$, (4)

then it is called a **fuzzy equality** on A.
Observe that for reflexive relations, condition (4) is equivalent with:

for all $x, y \in A$, $x \neq y$, $\rho(x, x) > \rho(x, y)$ and $\rho(x, x) > \rho(y, x)$.

Obviously, if (4) is not satisfied, i.e., if ρ is a fuzzy equivalence and not necessarily equality, then by reflexivity we have the following weaker condition.

$$\rho(x, x) \geq \rho(x, y) \text{ and } \rho(x, x) \geq \rho(y, x) \text{ for all } x, y \in A.$$

It is easy to show that if L is a lattice in which $\otimes$ is idempotent, then the last property is a consequence of symmetry and transitivity.
Let A = (A, F) be an algebra and L a complete residuated lattice. As it is known, a **fuzzy subalgebra** of A is any mapping $\mu : A \to L$ fulfilling the following:
For any operation f from F, $f : A^n \to A$, $n \in$ N, and all $x_1, \ldots, x_n \in A$, we have that

$$\otimes_{i=1}^{n} \mu(x_i) \leq \mu(f(x_1, \ldots, x_n)).$$

For a nullary operation (constant) $c \in F$, we require that

$$\mu(c) = 1, \tag{5}$$

where 1 is the top element in L.
A fuzzy relation $\rho : A^2 \to L$ on the underlying set A of an algebra A = (A, F) is said to be **compatible** with the operations, if for every n-ary operation $f \in F$ and for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$

$$\rho(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \geq \otimes_{i=1}^{n} \rho(x_i, y_i). \tag{6}$$

In particular, for $n = 1$, $\rho(f(x_1), f(y_1)) \geq \rho(x_1, y_1)$.
A reflexive, symmetric and transitive relation on A, which is compatible with fundamental operations on A, is a **fuzzy congruence** on this algebra. Obviously, special fuzzy congruences are also **compatible fuzzy equalities**, which additionally satisfy property (4).
Other notions from the basics of fuzzy sets that we use are either defined in the sequel where they appear, or they can be found in any introductory text about fuzziness.

Fuzzy compatible relations on fuzzy subalgebras

As it is known, fuzzy relations can be considered not only on crisp universes, but also on their fuzzy subsets.
Throughout the paper we consider fuzzy relations on fuzzy subalgebras of an algebra A.
Let A be a nonempty set, L a complete residuated lattice and $\mu : A \to L$ a fuzzy set on A. A fuzzy relation $\rho : A^2 \to L$ on A is said to be a **fuzzy relation on** μ if for all $x, y \in A$

$$\rho(x, y) \leq \mu(x) \otimes \mu(y). \tag{7}$$

As it is known, this condition is a generalization of the following crisp relational property: R is a binary relation on a subset M of A, if xRy implies $x, y \in M$.
Due to this condition, it is not possible to define reflexivity by (1). Therefore, we have the following definition.
A fuzzy relation ρ on a fuzzy set μ is **reflexive** if for all $x, y \in A$,

$$\rho(x, x) = \mu(x). \tag{8}$$

Reflexivity implies the following property of μ.

Proposition 1 *If ρ is a reflexive fuzzy relation on a fuzzy set μ on A, then for every $x, y \in A$,*
(i) $\mu(x) \otimes \mu(x) = \mu(x)$; (9)
(ii) $\rho(x, x) \geq \rho(x, y)$ *and* $\rho(x, x) \geq \rho(y, x)$. (10)

Proof. (i) is a consequence of (7) and (8), and (ii) follows by (i). □

Remark 1 If μ coincide with the characteristic function of A, i.e., if for all $x \in A$, $\mu(x) = 1$, then a fuzzy reflexive relation on μ is reflexive in the sense of (1): $\rho(x, x) = 1$, for all $x \in A$.

A fuzzy relation ρ on μ is symmetric and transitive if it fulfils the corresponding properties (2), (3) as a fuzzy relation on A.
A reflexive, symmetric and transitive relation ρ on μ is a **fuzzy equivalence** on this fuzzy set.
A fuzzy equivalence relation ρ on μ, satisfying also the strict inequality in (10), i.e., fulfilling for all $x, y \in A$:

$$\text{if } \rho(x, x) \neq 0, \text{ then } \rho(x, x) > \rho(x, y) \text{ and } \rho(x, x) > \rho(y, x), \tag{11}$$

is called a **fuzzy equality** relation on a fuzzy set μ.
In addition, a fuzzy relation ρ on μ is *compatible* with the operations on this fuzzy subalgebra if it fulfils the property (6). A compatible fuzzy equivalence on μ is a **fuzzy congruence** on this fuzzy subalgebra. A compatible fuzzy equality is called simply a **fuzzy equality** on μ.
Denote by fCon μ and fEqμ the collections of all fuzzy congruences and all compatible fuzzy equalities (respectively) on a fuzzy subalgebra μ of an algebra A. Obviously, the later is a sub-collection of the former, and both collections can be naturally ordered by componentwise order $\leq$, inherited from L.

Theorem 1 *The poset* (fCon μ, $\leq$) *is a complete lattice, while the poset* (fEq μ, $\leq$) *is a complete meet-semilattice, a semi-ideal in the former.*

Proof. First we prove that the poset (fCon μ, $\leq$) is a complete lattice by showing that it is closed under arbitrary infima and that it contains the greatest element.

Let $\{\rho_i \mid i \in I\}$ be an arbitrary nonempty subset of fCon μ and let

$$\varrho(x, y) := \bigwedge_{i\epsilon I} \rho_i(x, y).$$

We prove that $\rho \in$ fCon μ. Indeed,

$$\varrho(x, y) := \bigwedge_{i\epsilon I} \rho_i(x, y) \le \bigwedge_{i\epsilon I} \mu(x) \otimes \mu(y) = \mu(x) \otimes \mu(y),$$

which proves that ϱ fulfils (7), i.e., that it is a fuzzy relation on μ.
Now we show that ρ is reflexive, symmetric, transitive and compatible in the sense of (8), (2), (3) and (6) respectively:

$\varrho(x, x) = \bigwedge_{i\epsilon I} \rho_i(x, x) = \bigwedge_{i\epsilon I} \mu(x) = \mu(x);$

$\varrho(x, y) = \bigwedge_{i\epsilon I} \rho_i(x, y) = \bigwedge_{i\epsilon I} \rho_i(y, x) = \rho(y, x);$

$\varrho(x, y) = \bigwedge_{i\epsilon I} \rho_i(x, y) \ge \bigwedge_{i\epsilon I}(\rho_i(x, z) \otimes \rho_i(z, y))$

$\ge (\bigwedge_{i\epsilon I} \rho_i(x, z)) \otimes (\bigwedge_{i\epsilon I} \rho_i(z, y)) = \rho_i(x, z)) \otimes \rho_i(z, y));$

$\varrho(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) = \bigwedge_{i\epsilon I} \rho_i(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n))$

$\ge \bigwedge_{i\epsilon I}(\otimes_{j=1}^{n} \rho_i(x_j, y_j)) \ge \otimes_{j=1}^{n} (\bigwedge_{i\epsilon I} \rho_i(x_j, y_j)) = \otimes_{j=1}^{n} \varrho(x_j, y_j)$

for all $x, y, z, x_1, \ldots, x_n, y_1, \ldots, y_n \in A$ and for every non-nullary $f \in F$.
Thus, $\rho \in$ fCon μ.
Next we show that the poset (fCon μ, $\le$) contains the greatest element.
Indeed, the fuzzy relation $\rho^1 : A^2 \to L$, defined by

$$\rho^1(x, y) := \begin{cases} \mu(x), & x = y \\ \mu(x) \otimes \mu(y), & x \ne y. \end{cases}$$

is a fuzzy congruence on μ (the proof is straightforward), i.e., it belongs to (fCon μ, $\le$).
Obviously, for every $\rho \in$ fCon μ, we have that

$$\rho(x, y) \le \mu(x) \otimes \mu(y) = \rho^1(x, y).$$

Hence, ρ^1 is the top element in the poset (fCon μ, $\le$), which concludes the proof that it is a complete lattice.
It remains to prove that the poset (fEq μ, $\le$) of all compatible fuzzy equalities on μ is a meet-semilattice, being a semi-ideal in (fCon μ, $\le$). Since every compatible fuzzy equality is a fuzzy congruence on μ, the only nontrivial part of the proof is to show that the meet of an arbitrary nonempty collection $\{E_i \mid i \in I\}$ of compatible fuzzy equalities on μ satisfies the property (11). Let

$$\underline{E}(x, y) := \wedge_{i\epsilon I} E_i(x, y).$$

Now we have

$$\underline{E}(x, x) = \wedge_{i\epsilon I} E_i(x, x) = \mu(x) > \wedge_{i\epsilon I} E_i(x, y) = \underline{E}(x, y),$$

and similarly

$$\underline{E}(x, x) > \underline{E}(y, x).$$

To prove that (fEq μ, ≤) is a semi-ideal in (fCon μ, ≤) we mention that every fuzzy congruence on μ which is contained an arbitrary compatible fuzzy equality on μ is also a compatible fuzzy equality on the same fuzzy subalgebra. □

For the sake of completeness, we describe the bottom fuzzy relation in the foregoing lattice. The proof is straightforward.

Proposition 2 *The smallest fuzzy congruence (compatible fuzzy equality) on a fuzzy subalgebra μ of an algebra* A *is the mapping $E^0 : A^2 \to L$, given by*

$$E^0(x, y) := \begin{cases} \mu(x), & x = y \\ 0, & x \neq y. \end{cases}$$

Fuzzy equalities and fuzzy identities

The main reason for investigating fuzzy congruences is that we intend to use a fuzzy equality instead of the crisp one. In the crisp case, equality relation on a subalgebra B of an algebra A is a diagonal relation on B. It is obviously a congruence relation on B (it is also a particular congruence on A (see the book [18])). In the following, we develop an analogue fuzzy framework. Our main task is to use fuzzy equalities in order to deal with fuzzy identities which hold on fuzzy subalgebras and not necessarily on the whole (crip) algebra.

Recall that we talk about *fuzzy equalities*, meaning that these are also *compatible* with operations.

As before, we consider an algebra A = (A, F), and we have a complete residuated lattice L. Let μ be a fuzzy subalgebra of A and $E : A^2 \to L$ a fuzzy equality on μ. By the definition, E is reflexive in the sense of (8), it is symmetric, transitive and compatible with operations, and being defined on the fuzzy set μ, it fulfills also the property (7).

If t_1, t_2 are terms in the language of A, we consider the expression $E(t_1, t_2)$ as a **fuzzy identity with respect to** E, or (briefly) **fuzzy identity**, if E is fixed. Suppose that $x_1, \dots, x_n$ are variables appearing in terms t_1, t_2. We say that a fuzzy subalgebra μ of A **satisfies** a fuzzy identity $E(t_1, t_2)$ (or that a fuzzy identity is valid on a fuzzy subalgebra μ) if for all $x_1, \dots, x_n \in A$

$$\otimes_{i=1}^{n} \mu(x_i) \leq E(t_1, t_2). \quad (12)$$

The meaning of the formula (12) is usual: a fuzzy identity $E(t_1, t_2)$ is valid on a fuzzy subalgebra μ, if for any valuation replacing variables $x_1, \dots, x_n$ by elements from A, the inequality holds.

Proposition 3 *Let $\mu : A \to L$ be a fuzzy subalgebra of an algebra* A *and $E : A^2 \to L$ a fuzzy 7equality on μ. If μ satisfies a fuzzy identity $E(f, g)$, then also μ satisfies the identity $E_1(f, g)$, for every fuzzy equality E_1 on μ, such that $E \le E_1$.*

Proof. Suppose that μ satisfies a fuzzy identity $E(f, g)$ and that $x_1, \dots, x_n$ are variables appearing in f and g. Since $E_1 \ge E$ and $E(f, g) \ge \bigotimes_{i=1}^{n}\mu(x_i)$ by assumption, we have that also $E_1(f, g) \ge \bigotimes_{i=1}^{n}\mu(x_i)$. Hence, μ satisfies also the identity $E_1(f, g)$.
□

Lemma 1 *Let $\mu : A \to L$ be a fuzzy subalgebra of an algebra* A, *$\{E_i : A^2 \to L, i \in I\}$ a family of fuzzy equalities on μ, and f, g terms in the language of* A. *Now, if μ satisfies the identity $E_i(f, g)$ for every $i \in I$, then μ also satisfies the identity $E(f, g)$, where $E = \bigwedge_{i\epsilon I} E_i$.*

Proof. Assume that $x_1, \dots, x_n$ are variables appearing in f and g. Then we have the following.

$$E(f, g) = \bigwedge_{i\epsilon I} E_i(f, g) \ge \bigwedge_{i\epsilon I}\left(\bigotimes_{j=1}^{n} \mu(x_j)\right) = \bigotimes_{j=1}^{n} \mu(x_j).$$
□

Corollary 1 *If a fuzzy subalgebra μ of* A *satisfies the identity $E(f, g)$ for a fuzzy equality E, then there is the least fuzzy equality on μ, denoted by $E\mu(f,g)$, such that μ satisfies $E\mu(f,g)(f, g)$.*

Proof. Let $\{E_i \mid i \in I\}$ be a family of all fuzzy equalities on μ, such that for every $i \in I$, the identity $E_i(f, g)$ holds on μ. This family is not empty since by assumption it contains E. By Lemma 1, $E\mu(f,g) = \bigwedge_{i\epsilon I} E_i$. □

If the algebra A is identified with its characteristic function, then we get the following particular case of Corollary 1.
Recall that a residuated lattice L is said to be zero divisor free if
$x \otimes y = 0$ imply $x = 0$ or $y = 0$.

Theorem 2 *Let L be a zero divisor free residuated lattice and $\mu : A \to L$ a fuzzy subalgebra of an algebra* A, *such that $\mu(x) \neq 0$ for each $x \in A$. Then* A *satisfies the identity $f = g$ if and only if μ satisfies the identity $E(f, g)$ for every $E \in$ fEq μ.*

Proof. First we assume that the algebra A fulfils the crisp identity

$$f(x_1, \dots, x_n) = g(x_1, \dots, x_n).$$

Now, if $E \in$ fEq μ, then

$$E(f(x_1, \dots, x_n), g(y_1, \dots, y_n)) = \mu(f(x_1, \dots, x_n)) \ge \bigotimes_{i=1}^{n} \mu(x_i).$$

and μ satisfies the fuzzy identity $E(f, g)$.
Conversely, suppose that μ satisfies a fuzzy identity $E(f, g)$ i.e., that

$$E(f(x_1, \dots, x_n), g(y_1, \dots, y_n)) \geq \bigotimes_{i=1}^{n} \mu(x_i),$$

for all $E \in \mathrm{fEq}\,\mu$. The above formula holds also for the minimum fuzzy equality $E = E^0$ (Proposition 2). If $f(x_1, ..., x_n) \neq g(x_1, ..., x_n)$, then $E(f(x_1, ..., x_n), g(x_1, ..., x_n)) = 0$. Hence $\bigotimes_{i=1}^{n} \mu(x_i)=0$, which contradicts the assumption that L is zero divisor free, since, also by assumption, $\mu(x_i) \neq 0$ for all $x_1, \dots, x_n$.
Therefore, $f(x_1, ..., x_n) = g(x_1, ..., x_n)$. □

Next we present a more detailed connection of fuzzy and crisp identities in terms of fuzzy equalities and cut subalgebras. For this we assume that L is a particular residuated lattices for which the operation $\otimes$ coincides with the lattice meet, $\wedge$ (e.g., Heyting algebra). In such a case, every cut-set of a fuzzy subalgebra of μ is a crisp subalgebra of A, and every cut-relation of E is a congruence on the corresponding crisp cut-subalgebra (this is a well-known property of cuts, see e.g, [14, 21]).

Theorem 3 *Let μ be a fuzzy subalgebra of an algebra* A *and L a residuated lattice in which $\otimes$ coincides with $\wedge$. Let also f, g be terms in the language of* A. *Then the following hold.*
The cut subalgebra μ_p of μ for $p \in L$, $p \neq 0$ satisfies the crisp identity $f = g$ if and only if the cut-relation $(E\mu(f,g))_p$ of the least equality $E\mu(f,g)$ is the crisp equality on μ_p.

Proof. Assume that the subalgebra μ_p fulfils the crisp identity $f = g$.
We prove that in this case the smallest fuzzy equality $E\mu(f,g)$ coincides with E^0, defined in Proposition 2. Since for $x_1, ..., x_n \in \mu_p$ we have $f(x_1, ..., x_n) = g(x_1, ..., x_n)$, it is also true that

$$E^0(f(x_1, \dots, x_n), g(y_1, \dots, y_n)) = \mu(f(x_1, \dots, x_n)) \geq \wedge_{i=1}^{n} \mu(x_i).$$

Let $x, y \in \mu_p$ and $(x, y) \in \mathrm{E}_p^0$. By the definition $E^0(x, y) \geq p > 0$, hence $E(x, y) = \mu(x)$ i $x = y$. Therefore, E_p^0 is a crisp equality.
On the other hand, assume that $(E\mu(f,g))_p$ is a crisp equality. Let $x_1, ..., x_n \in \mu_p$. Then

$$E\mu(f,g)(f(x_1, ..., x_n), g(x_1, ..., x_n)) \geq \wedge_{i=1}^{n} \mu(x_i) \geq p,$$

hence $(f, g) \in (E\mu(f,g))_p$. Since $(E\mu(f,g))_p$ is a crisp equality, we finally have

$$f(x_1, ..., x_n) = g(x_1, ..., x_n).$$ □

Remark The last two proposition (Theorems 2 and 3) can be understood as follows: For an L-valued subalgebra μ of an algebra A, the minimal fuzzy equality $E\mu(f,g)$ expresses the "fuzziness" of the identity $f = g$ in μ. The smaller it is, more closer it is to the analogue crisp identity.

Connection to crisp algebras

Fuzzy relations on a fuzzy subalgebra of an algebra are also fuzzy relations on the algebra itself. In this section we analyze the above relational properties in this framework.

Thus, we have an algebra A = (A, F) and fuzzy relations on it, i.e., mappings from A^2 to a residuated lattice L.
We say that $\rho : A^2 \to L$ is a **weakly reflexive** relation on A, if

$\rho(x, x) \geq \rho(x, y)$, $\rho(x, x) \geq \rho(y, x)$ and (13)
$\rho(c, c) = 1$ for every constant $c \in F$. (14)

A fuzzy relation $\rho : A^2 \to L$ on an algebra A which is weakly reflexive, symmetric and transitive, is called a **weak fuzzy equivalence** on A. If, in addition, ρ fulfills also (11) i.e., the condition

$$\text{For all } x, y \in A,\ \rho(x, x) > \rho(x, y) \text{ and } \rho(x, x) > \rho(y, x),$$

then ρ is a **weak fuzzy equality** on A.
Obviously, particular cases of weak fuzzy equivalences are fuzzy equiv- alences (and equalities) satisfying classical reflexivity ($\rho(x, x) = 1$ for every $x \in A$). In the presence of weak reflexivity (13)(14), condition (4) is replaced by the weaker one, (11).
Observe that also the zero-relation ($\rho(x, x) = 0$ for every $x \in A$) is a weak congruence on an algebra without constants. We call it **trivial** weak congruence on this algebra.
Further, for compatible weak fuzzy equivalences fulfilling (6), we use the name **weak fuzzy congruences** on A. A subclass of weak fuzzy congruences are **compatible weak fuzzy equalities**.

Theorem 4 *If $\rho : A^2 \to L$ is a nontrivial weak fuzzy congruence on an algebra* A, *then the mapping $\mu_\rho : A \to L$, defined by*
$\mu_\rho(x) := \rho(x, x)$ (15)
is a fuzzy subalgebra of A.

Proof. Let f be an arbitrary fundamental n-ary operation on an algebra A and let $x_1, \ldots, x_n$ be elements from A. Then we have:

$$\mu_\rho(f(x_1, \ldots, x_n)) = \rho(f(x_1, \ldots, x_n), f(x_1, \ldots, x_n))$$

$$\geq \otimes_{i=1}^{n} \rho(x_i, x_i) = \otimes_{i=1}^{n} \mu_\rho(x_i),$$

and μ_ρ is a fuzzy subalgebra of A. □
Theorem 4 gives a link between fuzzy congruences on fuzzy subalgebras and weak fuzzy congruences on the whole algebra. For special lattices, these two notions coincide, as shown in the sequel.

Theorem 5 *Let L be a residuated lattice in which $\otimes$ coincides with $\wedge$. A weak fuzzy congruence $\rho : A^2 \to L$ on an algebra* A *is a fuzzy congruence on the fuzzy subalgebra μ_ρ of A. Conversely, a fuzzy congruence ρ on a fuzzy subalgebra μ of* A *is a weak fuzzy congruence on the whole algebra* A.

Proof. Let $\rho : A^2 \to L$ be a fuzzy congruence on an algebra A and μ_ρ a fuzzy subalgebra of A, defined by (15). According to (10), we have $\rho(x, x) \geq \rho(x, y)$ and $\rho(y, y) \geq \rho(x, y)$ for all $x, y \in A$, implying $\rho(x, y) \leq \mu_\rho(x) \wedge \mu_\rho(y)$, hence ρ is a fuzzy congruence on μ_ρ. Conversely, assume that ρ is a fuzzy congruence on a fuzzy subalgebra μ of A. Then we have $\rho(c, c) = \mu(c) = 1$, and (13) holds since $\wedge$ is idempotent. Thus, ρ is a weak fuzzy congruence on A.

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